Math 121 2.5 Higher-Order Derivatives

Objectives:
1) Find higher-order derivatives
2) Recognize different notation for higher-order derivatives.
3) Interpret the second derivative as the rate of change of the first derivative
4) Position, velocity and acceleration as derivatives.

To find higher-order derivatives, take the derivative of the derivative. How many times you do this depends on which higher-order derivative is requested:

Notation $f''(x) = \frac{d}{dx}(f'(x))$ 2nd derivative

$$y''(x) = \frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) = \frac{d^2}{dx^2}(f(x))$$

↑ remember: $\frac{d}{dx}$ is an operator which means "take the derivative"

If we do this twice $\frac{d}{dx}\left(\frac{d}{dx}(\text{stuff})\right)$ we get $\frac{d^2}{dx^2}(\text{stuff})$

$f'''(x) = \frac{d}{dx} f''(x)$ 3rd derivative

$$y'''(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)\right) = \frac{d^3}{dx^3}(f(x))$$

By the time we get to the fourth derivative, we stop using lots of primes and just put 4 in parentheses:

$f^{(4)}(x) = \frac{d}{dx} f'''(x)$ 4th derivative

$$y^{(4)}(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)\right)\right) = \frac{d^4}{dx^4}(f(x))$$

<u>CAUTION</u> $f^4(x)$ means $f(x)\cdot f(x)\cdot f(x)\cdot f(x)$ ← a big algebra problem but no calculus.

$f^{(4)}(x)$ means $\frac{d}{dx}(f'''(x))$ ← 4th derivative, repeated calculus.

① $f(x) = 2x^4 - 3x^3 + \frac{1}{2}x^2 + 17x - 11$

find the first five derivatives.

$f'(x) = 2\cdot 4x^3 - 3\cdot 3x^2 + \frac{1}{2}\cdot 2x + 17\cdot 1 - 0$

$\boxed{f'(x) = 8x^3 - 9x^2 + x + 17 \quad \text{first derivative}}$

$f''(x) = 8\cdot 3x^2 - 9\cdot 2x + 1 + 0$

$\boxed{f''(x) = 24x^2 - 18x + 1 \quad \text{second derivative}}$

$f'''(x) = 24\cdot 2x - 18\cdot 1 + 0$

$\boxed{f'''(x) = 48x - 18 \quad \text{third derivative}}$

$f^{(4)}(x) = 48\cdot 1 - 0$

$\boxed{f^{(4)}(x) = 48 \quad \text{fourth derivative}}$

$\boxed{f^{(5)}(x) = 0 \quad \text{fifth derivative}}$

② $g(x) = 25\sqrt{x}$

find the first three derivatives.

rewrite: $g(x) = 25x^{1/2}$

differentiate $g'(x) = 25\cdot\frac{1}{2}x^{\frac{1}{2}-1}$

$\boxed{g'(x) = \frac{25}{2}x^{-1/2} = \frac{25}{2x^{1/2}} = \frac{25}{2\sqrt{x}} \quad \text{first derivative}}$

differentiate again $g''(x) = \frac{25}{2}\cdot\left(\frac{-1}{2}\right)x^{-\frac{1}{2}-1}$

$\boxed{g''(x) = \frac{-25}{4}x^{-3/2} = \frac{-25}{4x^{3/2}} = \frac{-25}{4\sqrt{x^3}} \quad \text{2nd derivative}}$

differentiate again $g'''(x) = \frac{-25}{4}\cdot\frac{-3}{2}x^{-\frac{3}{2}-1}$

$\boxed{g'''(x) = +\frac{75}{8}x^{-5/2} = \frac{75}{8x^{5/2}} = \frac{75}{8\sqrt{x^5}} \quad \text{3rd derivative}}$

So... sometimes higher-order derivatives eventually become zeroes, and sometimes they don't.

(3) $f(x) = \dfrac{3x^2 - x + 5}{x}$

find $f''(x)$.

divide first using algebra

$$f(x) = \frac{3x^2}{x} - \frac{x}{x} + \frac{5}{x}$$

$$f(x) = 3x - 1 + 5x^{-1}$$

Differentiate

$$f'(x) = 3 - 0 + 5 \cdot (-1) x^{-2}$$

$$f'(x) = 3 - 5x^{-2}$$

Differentiate again

$$f''(x) = 0 - 5(-2)x^{-3}$$

$$\boxed{f''(x) = 10x^{-3} = \frac{10}{x^3}}$$

(4) $f(x) = \dfrac{3x+1}{3x-1}$

find $f'(x)$ and $f''(x)$.

Need Quotient Rule.

$$f'(x) = \frac{(3x-1) \cdot \frac{d}{dx}(3x+1) - (3x+1) \cdot \frac{d}{dx}(3x-1)}{(3x-1)^2}$$

$$= \frac{(3x-1) \cdot 3 - (3x+1) \cdot 3}{(3x-1)^2}$$

note $-(3x+1) \cdot 3$
$= -3(3x+1)$
is distribute both the 3 and the negative.

$$= \frac{9x - 3 - 9x - 3}{(3x-1)^2}$$

$$\boxed{f'(x) = \frac{-6}{(3x-1)^2}} = \frac{-6}{9x^2 - 6x + 1}$$

FOIL denominator

*Note: After 2.6 we'll have another way to find $\frac{d}{dx}(3x-1)^2$.

$$f''(x) = \frac{(9x^2 - 6x + 1) \cdot \frac{d}{dx}(-6) - (-6) \cdot \frac{d}{dx}(9x^2 - 6x + 1)}{[(3x-1)^2]^2}$$

$$f''(x) = \frac{(9x^2 - 6x + 1)\cdot 0 + 6\,(18x - 6 + 0)}{(3x-1)^{2\cdot 2}}$$

$$= \frac{0 + 108x - 36}{(3x-1)^4}$$

$$= \frac{108x - 36}{(3x-1)^4}$$

$$= \frac{36\,(3x-1)}{(3x-4)^4}$$

reduce $\frac{(3x-1)}{(3x-1)}$

$$= \frac{36}{(3x-4)^{4-1}}$$

$$\boxed{f''(x) = \frac{36}{(3x-4)^3}}$$

But what does $f''(x)$ mean?

Remember what $f'(x)$ means....

$f'(c)$ = slope of the tangent line to f at $x=c$

= instantaneous rate of change $\frac{\Delta y}{\Delta x}$ or $\frac{\Delta f}{\Delta x}$

Apply this logic to $f''(x)$ ....

$f''(c)$ = slope of the tangent line to f' at $x=c$

= instantaneous rate of change $\frac{\Delta f'}{\Delta x}$ — of the derivative!

We'll explore more about the graphs of f, f' and f'' in chapter 3

Position, Velocity and Acceleration

Here's a concrete example of $f(x)$, $f'(x)$ and $f''(x)$.

Replace f by
x by t } we get (t).

$s(t)$ represents the location <u>ON A LINE</u> of a moving object at time time t. This is called a <u>rectilinear position function.</u>

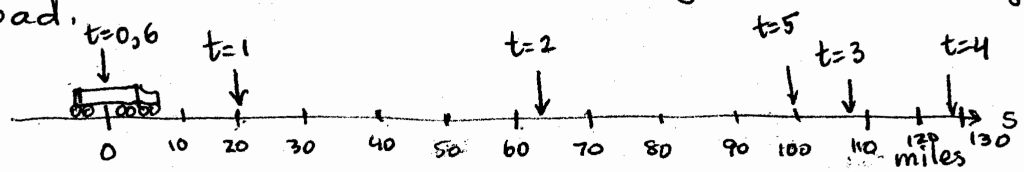
5 Example: $s(t) = 24t^2 - 4t^3$ for $0 \le t \le 6$.

where distance s is in miles
and time t is in hours

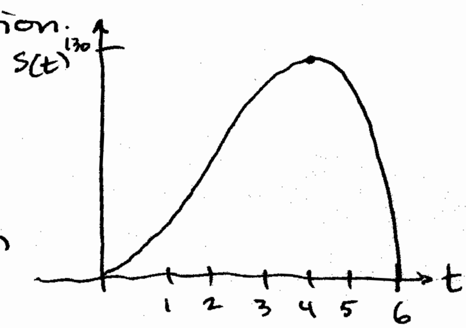
We don't allow t to be negative because -2 hours doesn't make sense!

We don't allow t to be greater than 6 because our function is a polynomial (to make the calculus easy) which doesn't give realistic values after 6 hours.

In this example, we have a large truck driving on a straight road.



t	$s(t)$	
0	0	at time 0, truck is at position 0 miles
1	20	
2	64	
3	108	
4	128	← at $t=4$, the truck turns around!
5	100	← at $t=5$, 100 miles from starting position
6	0	← at $t=6$, back to starting position.



Graph of this position function

Velocity $v(t)$

The $\underbrace{\text{rate of change}}_{\text{take a derivative}}$ of the $\underbrace{\text{position function}}_{\text{of } s(t)}$ $= s'(t)$.
$= v(t)$

$$s'(t) = \frac{d}{dt}(24t^2 - 4t^3)$$

notice we use $\frac{d}{dt}$ instead of $\frac{d}{dx}$!

$$v(t) = s'(t) = 48t - 12t^2 = \frac{\Delta \text{position (miles)}}{\Delta \text{time (hours)}} = \text{velocity } v(t) \text{ in mph}$$

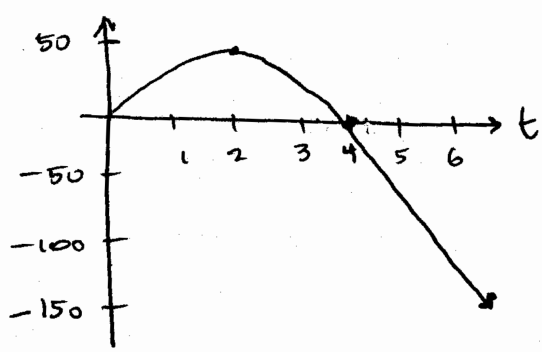
t	$v(t) = s'(t)$
0	0 mph
1	36 mph
2	48 mph
3	36 mph
4	0 mph
5	−60 mph
6	−144 mph

Negative! Yes!

Velocity is a vector, meaning it has
- numerical value
- direction – the direction is in the sign!

So −60 mph means
- going 60 mph
- going back the way it came (to the left).

Graph of velocity function.



Acceleration $a(t)$

The $\underbrace{\text{rate of change}}_{\text{take derivative of}}$ of the $\underbrace{\text{velocity function}}_{v(t)=s'(t)}$ $= v'(t) = s''(t)$
$= a(t)$

$$a(t) = s''(t) = v'(t) = \frac{d}{dt}(48t - 12t^2)$$

$$a(t) = 48 - 24t = \frac{\Delta \text{velocity}}{\Delta \text{time}} \frac{\text{mph}}{\text{hr}} = \text{acceleration } a(t)$$

in miles per hour per hour
or miles / hr^2.

In colloquial use, acceleration is how fast the speed changes, as in... "my race car goes from 0mph to 60 mph in 3 seconds!" (very good acceleration)

or "my clunker goes from 0mph to 40 mph in 20 seconds." (very poor acceleration)

t	$a(t) = v'(t) = s''(t)$	
0	48	m/hr^2
1	24	m/hr^2
2	0	m/hr^2
3	-24	
4	-48	
5	-72	
6	-96	

All three functions tell us instantaneous measurements, which are valid only at that single moment.

	0	1	2	3	4	5	6
$s(t)$	0	20	64	108	128	100	0
$v(t)$	0	36	48	36	0	-60	-144
$a(t)$	48	24	0	-24	-48	-72	-96
	At the starting location. Not moving (yet.) Speed increasing	20 miles away 36 mph speed increasing	64 miles away 48 mph speed constant	108 mi away 36 mph speed decreasing	128 mi away 0mph speed increasing (to left)	100 mi away 60 mph to left speed increasing (to left)	At starting location again 144 mph to left speed increasing (to left)

If velocity and acceleration are

- both positive $\Rightarrow$ speeding up to the right
- both negative $\Rightarrow$ speeding up to the left
- $v(t)$ positive, $a(t)$ negative $\Big\} \Rightarrow$ slowing down to right
- $v(t)$ negative, $a(t)$ positive $\Big\} \Rightarrow$ slowing down to left.

6 During a recent Hepatitis A epidemic in San Diego...

a) Early news reports said:

"The rate of increase in the number of cases this week was greater than last week."

$f' > 0 \rightarrow$ People were still being infected. (# cases increasing) (rate of change is positive)

$f'' > 0 \rightarrow$ The rate of infection was increasing. (more people sickened this week than last week) (rate of change increasing) (2nd derivative positive)

b) Later news reports said:

"The rate of increase in the number of cases was less than last week."

$f' > 0 \longrightarrow$ People were still being infected. (# cases increasing) rate of change is positive)

$f'' < 0 \longrightarrow$ The rate of infection was decreasing (fewer people sickened this week than last). (rate of change decreasing) (2nd derivative is negative)

c) Final reports said

"The number of cases decreased this week."

$f' < 0 \longrightarrow$ More people got well than got sick. # cases decreased rate of change negative derivative negative

f'' not reported. $\rightarrow$ (Second derivative was not reported.)